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Global Approximation Theorems for Some Exponential-type Operators

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1. INTRODUCTION

Let $f(x)$ be a continuous real-valued function on the interval $D := [A, B] \cap (-\infty, \infty)$, in symbols: $f \in C[A, B]$. We use the following notation:

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h) \qquad (x \in D_h),$$

$$\|f\| = \sup_{x \in D} |f(x)|,$$

$$\omega_2(f; \delta) = \sup_{0<h\leqslant\delta} \sup_{x \in D_h} |\Delta_h^2 f(x)|,$$

$$\mathrm{Lip}_2\, \alpha = \{f \in C[A, B]; \omega_2(f; \delta) = O(\delta^\alpha), \delta \to 0_+\},$$

where $D_h := [A+h, B-h] \cap (-\infty, \infty)$.

Let us consider the operators

$$L_n(f; x) = \int_A^B W(n, x, u) f(u)\, du \qquad (n \geqslant 1),$$

where $W(n, x, u) \geqslant 0$ is a function on D. We say that $L_n(f; x)$ is an exponential-type operator if

$$\int_A^B W(n, x, u)\, du = 1 \tag{1}$$

and

$$\frac{\partial}{\partial x} W(n, x, u) = \frac{n}{\phi(x)} W(n, x, u)(u - x), \tag{2}$$

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where $\phi(x)$ is a polynomial of degree $\leqslant 2$, $\phi(x)>0$ on (A, B) and $\phi(A)=0$, $\phi(B)=0$ if $A,\ B \neq \pm\infty$. Such exponential-type operators were first introduced by May (cf. [9]).

$L_n(f; x)$ is a positive operator since $W(n, x, u) \geqslant 0$. By (1) and (2) we have

$$\int_A^B W(n, x, u) u\, du = x \qquad \text{for} \quad x \in D.$$

Hence $L_n(f; x)$ preserves linear functions.

Let us write

$$A_m(n, x) = n^m \int_A^B W(n, x, u)(u-x)^m\, du.$$

Using (2), we have (see [9])

$$A_{m+1}(n, x) = nm\phi(x) A_{m-1}(n, x) + \phi(x) \frac{d}{dx} A_m(n, x)$$

and by simple calculations,

$$\begin{aligned} &A_0(n, x) = 1, \qquad && A_1(n, x) = 0, \\ &A_2(n, x) = n\phi(x), \qquad && A_3(n, x) = n\phi(x)\, \phi'(x), \\ &A_4(n, x) = 3n^2\phi(x)^2 + n\phi(x)[\phi'(x)^2 + \phi(x)\, \phi''(x)]. && \end{aligned} \tag{3}$$

Many authors have considered global approximation theorems for specific exponential-type operators (cf. [3, 6], etc.). In this paper we try to generalize these theorems using an elementary method. Under some conditions upon $\phi(x)$ we prove a theorem for $L_n(f; x)$ which includes results for Bernstein polynomials and for Gauss–Weierstrass, Szász–Mirakjan, Baskakov and other operators.

We impose on $\phi(x)$ the condition that it is a polynomial of degree $\leqslant 2$ without a double zero and satisfies

$$I := \frac{n^2}{\phi(x)^2} \int_\alpha^\beta \left[\frac{\phi'(x)}{n} - u + x\right] (u-x)^3\, W(n, x, u)\, du \leqslant M, \tag{4}$$

where $\alpha = \min(x, x + \phi'(x)/n)$ and $\beta = \max(x, x + \phi'(x)/n)$. Throughout the paper let M be an absolute constant independent of n and x. Then we have the following theorem.

THEOREM 1. *Let $\phi(x)$ satisfy the above condition. Then for* $0 < \alpha < 2$ *the following statements are equivalent*:

$$\text{(I)} \quad f \in \operatorname{Lip}_2 \alpha,$$

$$\text{(II)} \quad |L_n(f; x) - f(x)| \leqslant M \left[\frac{\phi(x)}{n}\right]^{\alpha/2} \qquad (n \geqslant 1, x \in D).$$

We denote this result by the notation

$$\text{G.App.}[L_n] = [\{f \mid f \in \operatorname{Lip}_2 \alpha\}, n^{-\alpha/2}, \phi(x), D].$$

The method for proving the direct part (I) $\Rightarrow$ (II) of this theorem is the standard procedure using a Jackson-type inequality, the Steklov means and appropriate estimates of the moments of the operators. For proving the inverve part (II) $\Rightarrow$ (I) we use the elementary method which was introduced by Berens and Lorentz [5] and was further developed in [2] and [4]. Since the elementary method fails for the saturation case $\alpha = 2$ in the inverse part, we only consider the nonoptimal case $0 < \alpha < 2$.

2. PROOF OF THE DIRECT PART

In this section we prove the direct part of Theorem 1.

Let us introduce $F(x)$ by the continuous extension of $f(x)$ onto $(-\infty, \infty)$

$$\begin{aligned} F(x) &= f(x), & A \leqslant x \leqslant B, \\ &= f(2A - x), & 2A - B \leqslant x \leqslant A, \end{aligned}$$

where $F(x)$ is $2(B - A)$ periodic when A and B are finite (see [10, p. 122]). Then for the Steklov means

$$f_h(x) = \frac{1}{h^2} \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} F(x + s + t)\, ds\, dt \qquad (h > 0)$$

we have the following estimates (cf. [5, 10])

$$\|f - f_h\| \leqslant \frac{5}{2} \omega_2(f; h), \qquad \|f_h''\| \leqslant \frac{5}{h^2} \omega_2(f; h). \tag{5}$$

Now let us write

$$C^2[A, B] = \{f \in C[A, B]; f'' \in C[A, B]\},$$

then we have the following inequality.

PROPOSITION (Jackson-type inequality). *For $g \in C^2[A,B]$ we have*

$$|L_n(g;x) - g(x)| \leqslant M \| g'' \| \frac{\phi(x)}{n} \qquad (n \geqslant 1, x \in D).$$

Proof. In view of

$$g(t) - g(x) = (t-x)\, g'(x) + \int_x^t \int_x^s g''(u)\, du\, ds,$$

there follows

$$\begin{aligned} L_n(g;x) - g(x) &= L_n(g(t) - g(x); x) \\ &= L_n\left((t-x)\, g'(x) + \int_x^t \int_x^s g''(u)\, du\, ds; x\right) \\ &= L_n\left(\int_x^t \int_x^s g''(u)\, du\, ds; x\right). \end{aligned}$$

From the positivity of $L_n(f;x)$ we get

$$\begin{aligned} |L_n(g;x) - g(x)| &\leqslant L_n\left(\left|\int_x^t \int_x^s g''(u)\, du\, ds\right|; x\right) \\ &\leqslant L_n\left(\| g'' \| \int_x^t \int_x^s du\, ds; x\right) = \| g'' \| L_n\left(\frac{(t-x)^2}{2}; x\right). \end{aligned}$$

Hence the proof is completed by (3).

Now we verify the direct part (I) $\Rightarrow$ (II). Since $f \in C[A,B]$ is approximated by $\{e^{kx}\}$ and $L_n(e^{ku};x) = e^{kx}$ holds for $x = A, B$ $(A, B \neq \pm \infty)$ by the recursion relation of $A_m(n,x)$, we have $L_n(f;x) = f(x)$ $(x = A, B)$ easily. Therefore we consider the proof for $x \in (A,B)$. Since $L_n(f;x)$ is a bounded operator, we have for $h \leqslant \sqrt{\phi(x)}$ by the Proposition

$$\begin{aligned} |L_n(f;x) - f(x)| &\leqslant |L_n(f - f_h; x)| + |L_n(f_h; x) - f_h(x)| \\ &\quad + |f_h(x) - f(x)| \\ &\leqslant M \|f - f_h\| + M \|f_h''\| \frac{\phi(x)}{n} + \|f_h - f\|. \end{aligned}$$

Using (5), we obtain

$$|L_n(f;x) - f(x)| \leqslant M\omega_2(f;h)[1 + \phi(x)/nh^2].$$

By the assumption $f \in \mathrm{Lip}_2\,\alpha$, setting $h = \sqrt{\phi(x)/n}$, we have

$$|L_n(f; x) - f(x)| \leqslant M\omega_2\left(f; \left[\frac{\phi(x)}{n}\right]^{1/2}\right) \leqslant M\left[\frac{\phi(x)}{n}\right]^{\alpha/2}.$$

Remark 1. Obviously, the direct part holds without condition (4).

3. Proof of the Inverse Part

In order to prove the inverse part, by the idea of Berens and Lorentz [5], it is sufficient to show that for $f \in C[A, B]$ and $0 < \alpha < 2$

$$\omega_2(f; h) \leqslant M\left[\delta^\alpha + \left(\frac{h}{\delta}\right)^2 \omega_2(f; \delta)\right], \tag{6}$$

where $0 < h < \min((B-A)/4, 1)$ and $0 < \delta < \min(\sup_x \sqrt{\phi(x)}, 1)$. To obtain (6), we need the following two lemmas.

Lemma 1. *Let $\phi(x)$ satisfy condition* (4). *For $g \in C^2[A, B]$ there holds*

$$|[L_n(g; x)]''| \leqslant M\,\|g''\| \qquad (n \geqslant 1, x \in (A, B)).$$

Proof. By Taylor expansion of g, we have

$$\begin{aligned}
\frac{d^2}{dx^2} L_n(g; x) &= \int_A^B \left[\frac{\partial^2}{\partial x^2} W(n, x, u)\right] g(u)\,du \\
&= \int_A^B \left[\frac{\partial^2}{\partial x^2} W(n, x, u)\right] \\
&\qquad \times \{g(x) + (u-x)\,g'(x) + (u-x)^2\,g''(\xi)\}\,du \\
&= g(x)\int_A^B \left[\frac{\partial^2}{\partial x^2} W(n, x, u)\right] du \\
&\qquad + g'(x)\int_A^B \left[\frac{\partial^2}{\partial x^2} W(n, x, u)\right](u-x)\,du \\
&\qquad + \int_A^B \left[\frac{\partial^2}{\partial x^2} W(n, x, u)\right](u-x)^2\,g''(\xi)\,du.
\end{aligned}$$

Since from (3)

$$\int_A^B \left[\frac{\partial^2}{\partial x^2} W(n, x, u)\right](u-x)^i\,du = 0 \qquad \text{for} \quad i = 0, 1,$$

we have

$$\frac{d^2}{dx^2} L_n(g; x) = \int_A^B \left[\frac{\partial^2}{\partial x^2} W(n, x, u)\right] (u-x)^2 g''(\xi)\, du,$$

and hence

$$\left|\frac{d^2}{dx^2} L_n(g; x)\right| \leqslant \|g''\| \int_A^B \left|\frac{\partial^2}{\partial x^2} W(n, x, u)\right| (u-x)^2\, du.$$

Thus it is sufficient to show that

$$J := \int_A^B \left|\frac{\partial^2}{\partial x^2} W(n, x, u)\right| (u-x)^2\, du \leqslant M. \tag{7}$$

In view of (3) and

$$\frac{\partial^2}{\partial x^2} W(n, x, u) = \frac{n}{\phi(x)} \left[-\frac{\phi'(x)}{\phi(x)} (u-x) + \frac{n}{\phi(x)} (u-x)^2 - 1\right] W(n, x, u), \tag{8}$$

we have

$$\begin{aligned} J &= \int_A^B \frac{n}{\phi(x)} \left|-\frac{\phi'(x)}{\phi(x)} (u-x) + \frac{n}{\phi(x)} (u-x)^2 - 1\right| \\ &\quad \times (u-x)^2 W(n, x, u)\, du \\ &\leqslant \frac{n}{\phi(x)} \int_A^B \left|-\frac{\phi'(x)}{\phi(x)} (u-x) + \frac{n}{\phi(x)} (u-x)^2\right| \\ &\quad \times (u-x)^2 W(n, x, u)\, du + 1. \end{aligned}$$

Since

$$-\frac{\phi'(x)}{\phi(x)} (u-x) + \frac{n}{\phi(x)} (u-x)^2 = \frac{n}{\phi(x)} (u-x) \left(u - x - \frac{\phi'(x)}{n}\right),$$

taking $\alpha = \min(x, x + \phi'(x)/n)$ and $\beta = \max(x, x + \phi'(x)/n)$, and by (3) we have

$$\begin{aligned} J \leqslant{} & \frac{n}{\phi(x)} \int_A^\alpha \left[-\frac{\phi'(x)}{\phi(x)} (u-x)^3 + \frac{n}{\phi(x)} (u-x)^4\right] W(n, x, u)\, du \\ & + \frac{n}{\phi(x)} \int_\alpha^\beta \left[\frac{\phi'(x)}{\phi(x)} (u-x)^3 - \frac{n}{\phi(x)} (u-x)^4\right] W(n, x, u)\, du \\ & + \frac{n}{\phi(x)} \int_\beta^B \left[-\frac{\phi'(x)}{\phi(x)} (u-x)^3 + \frac{n}{\phi(x)} (u-x)^4\right] W(n, x, u)\, du + 1 \end{aligned}$$

$$= \frac{n^2}{\phi(x)^2} \int_A^B W(n, x, u)(u-x)^4 \, du - \frac{n\phi'(x)}{\phi(x)^2} \int_A^B W(n, x, u)(u-x)^3 \, du$$
$$+ \frac{2n^2}{\phi(x)^2} \int_\alpha^\beta \left[\frac{\phi'(x)}{n}(u-x)^3 - (u-x)^4 \right] W(n, x, u) \, du + 1$$
$$= 4 + \frac{\phi''(x)}{n} + 2I.$$

Hence by (4) we obtain (7). This completes the proof.

Remark 2. By (3) we have the following estimation

$$I \leqslant \frac{n^2}{\phi(x)^2} \int_\alpha^\beta \frac{\phi'(x)^2}{n^2} (u-x)^2 \, W(n, x, u) \, du$$
$$+ \frac{n^2}{\phi(x)^2} \int_\alpha^\beta (u-x)^4 \, W(n, x, u) \, du$$
$$\leqslant \frac{\phi'(x)^2}{\phi(x)^2} \int_A^B W(n, x, u)(u-x)^2 \, du$$
$$+ \frac{n^2}{\phi(x)^2} \int_A^B W(n, x, u)(u-x)^4 \, du$$
$$= \frac{2\phi'(x)^2}{n\phi(x)} + 3 + \frac{\phi''(x)}{n}.$$

Thus (4) holds for those x such that (see also Section 4).

$$[\phi'(x)]^2 \leqslant Mn\phi(x). \tag{9}$$

LEMMA 2. *Let $\phi(x)$ satisfy condition* (4). *Then we have for $x \in D_h$ ($D_h = [A+h, B-h] \cap (-\infty, \infty)$)*

$$\int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \frac{ds \, dt}{\phi(x+s+t)} \leqslant \frac{Mh^2}{m(x, h)}, \tag{10}$$

where

$$m(x, h) = \max\{\phi(x-h), \phi(x), \phi(x+h)\}$$

and

$$0 < h < \min\{(B-A)/4, 1\}.$$

Proof. Ismail and May [8] have shown that the family of exponential-type operators essentially consists of 6 different types of operators according to $\phi(x) \in \{1, x, x(1-x), x(1+x), x^2, x^2+1\}$. In view of the conditions on $\phi(x)$, we only prove (10) for the five normalized cases $\{\phi(x)\} = \{1, x, x(1-x), x(1+x), x^2+1\}$. The case $\phi(x) = 1$ is trivial. For $\phi(x) = x(1-x)$

see [4]. For $\phi(x)=x$, $x(1+x)$ see [2, p. 138]. Thus there remains the case $\phi(x)=x^2+1$.

In this case we have for $x \in [0, h]$

$$\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} \frac{ds\,dt}{\phi(x+s+t)} \leqslant h^2 \leqslant \frac{Mh^2}{(x+h)^2+1} = \frac{Mh^2}{m(x,h)}.$$

Since $4hx \leqslant M(x-h)^2+M$ for $M \geqslant 5$ and $0<h<1$, we obtain the following estimate for $x \in [h, \infty)$

$$\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} \frac{ds\,dt}{\phi(x+s+t)}$$

$$\leqslant \frac{h^2}{\phi(x-h)} = \left[1+\frac{4hx}{(x-h)^2+1}\right]\frac{h^2}{(x+h)^2+1} \leqslant \frac{(1+M)\,h^2}{m(x,h)}.$$

Analogously we get the inequality (10) for $x \in (-\infty, 0]$. Hence the proof of Lemma 2 is complete.

Remark 3. Since the assertion of Lemma 2 does not hold for $\phi(x)=x^2$, we exclude $\phi(x)=x^2$, as the elementary method of proof fails just in this case (cf. [1]).

And now, we prove the inverse part (II) $\Rightarrow$ (I). By the assumption (II) we have for $x \in D_h$

$$\begin{aligned} |\Delta_h f(x)| &= |f(x+h)-2f(x)+f(x-h)| \\ &\leqslant |f(x+h)-L_n(f; x+h)| + 2\,|f(x)-L_n(f; x)| \\ &\quad + |f(x-h)-L_n(f; x-h)| \\ &\quad + |L_n(f; x+h)-2L_n(f; x)+L_n(f; x-h)| \\ &\leqslant M\left\{\left[\frac{\phi(x+h)}{n}\right]^{\alpha/2} + 2\left[\frac{\phi(x)}{n}\right]^{\alpha/2} + \left[\frac{\phi(x-h)}{n}\right]^{\alpha/2}\right\} \\ &\quad + \left|\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} [L_n(f; x+s+t)]''\,ds\,dt\right|. \end{aligned}$$

Thus we have to estimate for $0<\delta<\min(\sup_x \sqrt{\phi(x)}, 1)$

$$[L_n(f; x)]'' = [L_n(f-f_\delta; x)]'' + [L_n(f_\delta; x)]''. \tag{11}$$

Using (8), we have

$$|[L_n(f-f_\delta; x)]''|$$

$$\leqslant \frac{n}{\phi(x)}\,\|f-f_\delta\| \int_A^B \left|-\frac{\phi'(x)}{\phi(x)}(u-x)+\frac{n}{\phi(x)}(u-x)^2-1\right| W(n,x,u)\,du$$

$$\leqslant \frac{n}{\phi(x)} \|f - f_\delta\| \left[\frac{|\phi'(x)|}{\phi(x)} \int_A^B W(n, x, u)\, |u - x|\, du \right.$$

$$\left. + \frac{n}{\phi(x)} \int_A^B W(n, x, u)(u - x)^2\, du + \int_A^B W(n, x, u)\, du \right],$$

and by (3)

$$|[L_n(f - f_\delta; x)]''|$$

$$\leqslant \frac{n}{\phi(x)} \|f - f_\delta\| \left\{ \frac{|\phi'(x)|}{\phi(x)} \int_A^B W(n, x, u)\, |u - x|\, du + 2 \right\}. \qquad (12)$$

To estimate (12), we prove the following inequality

$$\frac{|\phi'(x)|}{\phi(x)} \int_A^B W(n, x, u)\, |u - x|\, du \leqslant M. \qquad (13)$$

When $\phi(x)$ is constant, (13) is evident. Next we consider the case that $\phi(x)$ has zero points. From $L_n((t - x); x) = 0$ we have

$$\int_A^B W(n, x, u)\, |u - x|\, du$$

$$= \int_A^x W(n, x, u)(x - u)\, du + \int_x^B W(n, x, u)(u - x)\, du$$

$$= 2 \int_A^x W(n, x, u)(x - u)\, du \leqslant 2(x - A),$$

and similarly

$$\int_A^B W(n, x, u)\, |u - x|\, du \leqslant 2(B - x).$$

Since A and B are zero points of $\phi(x)$, we obtain

$$\frac{|\phi'(x)|}{\phi(x)} \int_A^B W(n, x, u)\, |u - x|\, du$$

$$\leqslant \frac{|\phi'(x)|}{\phi(x)} \min(2(x - A), 2(B - x)) \leqslant M.$$

For the remaining case that a quadratic polynomial $\phi(x)$ has no zero points, we have (cf. Section 4)

$$\frac{|\phi'(x)|}{\phi(x)} \int_A^B W(n, x, u)\,|u - x|\,du$$
$$\leqslant \frac{|\phi'(x)|}{\phi(x)} \left\{ \int_A^B W(n, x, u)(u - x)^2\,du \right\}^{1/2}$$
$$= \frac{|\phi'(x)|}{\phi(x)} \left\{ \frac{\phi(x)}{n} \right\}^{1/2} = \frac{|\phi'(x)|}{\sqrt{n\phi(x)}} \leqslant M.$$

Hence from (13) and (5) we have

$$|[L_n(f - f_\delta; x)]''| \leqslant M \frac{n}{\phi(x)} \|f - f_\delta\| \leqslant M \frac{n}{\phi(x)} \omega_2(f; \delta). \tag{14}$$

By the boundedness of $L_n(f; x)$ and Lemma 1, we obtain the estimation of the second term of (11)

$$|[L_n(f_\delta; x)]''| \leqslant M \|f_\delta''\| \leqslant M \frac{1}{\delta^2} \omega_2(f; \delta). \tag{15}$$

Combining (14) and (15), we have

$$|[L_n(f; x)]''| \leqslant M\omega_2(f; \delta) \left[\frac{n}{\phi(x)} + \frac{1}{\delta^2} \right].$$

Then, by $m(x, h) = \max\{\phi(x - h), \phi(x), \phi(x + h)\}$ and Lemma 2, we obtain for $x \in D_h$

$$|\Delta_h f(x)| \leqslant M \left\{ \left[\frac{\phi(x + h)}{n} \right]^{\alpha/2} + 2 \left[\frac{\phi(x)}{n} \right]^{\alpha/2} + \left[\frac{\phi(x - h)}{n} \right]^{\alpha/2} \right\}$$
$$+ M\omega_2(f; \delta) \left[n \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \frac{ds\,dt}{\phi(x + s + t)} + \left(\frac{h}{\delta} \right)^2 \right]$$
$$\leqslant M \left\{ \left[\frac{m(x, h)}{n} \right]^{\alpha/2} + \omega_2(f; \delta) \left[\frac{nh^2}{m(x, h)} + \left(\frac{h}{\delta} \right)^2 \right] \right\}.$$

Choosing n such that

$$\sqrt{\frac{m(x, h)}{n}} \leqslant \delta < \sqrt{\frac{m(x, h)}{n - 1}} \leqslant \sqrt{2} \sqrt{\frac{m(x, h)}{n}},$$

we have

$$|\Delta_h f(x)| \leqslant M \left[\delta^\alpha + \omega_2(f; \delta) \left(\frac{h}{\delta} \right)^2 \right].$$

Hence we obtain (6) and by induction

$$\omega_2(f; h) \leqslant Mh^{\alpha},$$

which implies $f \in \mathrm{Lip}_2\, \alpha$. This completes the proof of the inverse part.

4. Applications

We apply Theorem 1 to the five normalized types of operators for $\{\phi(x)\} = \{1, x, x(1-x), x(1+x), x^2+1\}$. Especially for these $\phi(x)$ we consider condition (4). Since $\phi(x) = 1$ and x^2+1 satisfy (9) for all $x \in D$, condition (4) holds for $\phi(x) = 1$ and x^2+1 by Remark 2. For the remaining cases $\phi(x) = x$, $x(1-x)$, $x(1+x)$ condition (9) holds whenever $\phi(x) \geqslant a/n$ for some (small) $\alpha > 0$. Hence condition (4) needs only to be tested for three cases of $\phi(x)$ in a small neighborhood of the zero points of $\phi(x)$.

In these three cases the operator $L_n(f; x)$ may be represented as a sum

$$L_n(f; x) = \sum_{k/n \in D} f\left(\frac{k}{n}\right) p_{k,n}(x) \qquad (n \geqslant 1, D = [A, B] \cap (-\infty, \infty)).$$

Then condition (4) becomes

$$I_k := n^2 \left(\frac{k}{n} - x\right)^3 \left(x - \frac{k-1}{n}\right) p_{k,n}(x)\, \phi(x)^{-2} \leqslant M$$

for $x \leqslant k/n \leqslant x + 1/n$ and needs only to be tested for $k = 0, 1$ (concerning a neighborhood of $x = 1$ in the case $\phi(x) = x(1-x)$, we may use analogous arguments). We have for $0 \leqslant x \leqslant 1/n$

$$I_0 \leqslant n^2 x^3 \left(x + \frac{1}{n}\right) p_{0,n}(x)\, \phi(x)^{-2} \leqslant 2\left(\frac{x}{\phi(x)}\right)^2 p_{0,n}(x),$$

$$I_1 = n^2 \left(\frac{1}{n} - x\right)^3 x p_{1,n}(x)\, \phi(x)^{-2} \leqslant \frac{x p_{1,n}(x)}{n\phi^2(x)}.$$

Thus (4) follows from

$$\max_{0 \leqslant x \leqslant 1/n} \left\{\frac{x^2 p_{0,n}(x)}{\phi^2(x)}, \frac{x p_{1,n}(x)}{n\phi^2(x)}\right\} \leqslant M. \tag{16}$$

Therefore instead of (4) we shall check (16) for the three cases $\phi(x) = x$, $x(1-x)$, $x(1+x)$.

4.1. *Gauss–Weierstrass Operators*

We define the Gauss–Weierstrass operator $G_n(f; x)$ as follows.

$$G_n(f; x) = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{n(u-x)^2}{2}\right\} f(u)\, du \qquad (x \in (-\infty, \infty)).$$

Then the kernel of this operator is

$$W(n, x, u) = \sqrt{\frac{n}{2\pi}} \exp\left\{-\frac{n(u-x)^2}{2}\right\}.$$

From Remark 2 and the above we have the corresponding theorem for $G_n(f; x)$.

THEOREM 2.

$$\text{G.App.}[G_n] = [\{f \mid f \in \text{Lip}_2\, \alpha\}, n^{-\alpha/2}, 1, (-\infty, \infty)].$$

4.2. *Szász–Mirakjan Operators*

The Szász–Mirakjan operator $S_n(f; x)$ is defined as follows.

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \qquad (x \in [0, \infty)),$$

$$W(n, x, u) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \delta\left(u - \frac{k}{n}\right).$$

THEOREM 3. (Becker [3]).

$$\text{G.App.}[S_n] = [\{f \mid f \in \text{Lip}_2\, \alpha\}, n^{-\alpha/2}, x, [0, \infty)].$$

Proof. In view of the above we only test that $\phi(x) = x$ and $p_{k,n}(x) = (nx)^k/k!$ satisfy (16) for $0 \leqslant x \leqslant 1/n$. From $p_{0,n}(x) = 1$ and $p_{1,n}(x) = nx$, we obtain inequality (16) immediately.

4.3. *Bernstein Polynomials.*

The Bernstein polynomials are defined by

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^n f\left(\frac{k}{n}\right) \qquad (x \in [0, 1]),$$

$$W(n, x, u) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \delta\left(u - \frac{k}{n}\right).$$

Then we have the following theorem.

THEOREM 4. (Berens and Lorentz [5]).

$$\text{G.App.}[B_n] = [\{f \mid f \in \text{Lip}_2\, \alpha\}, n^{-\alpha/2}, x(1-x), [0, 1]].$$

Proof. Since $\phi(x) = x(1-x)$ and $p_{k,n}(x) = \binom{n}{k} x^k(1-x)^{n-k}$, we have

$$p_{0,n}(x) = (1-x)^n, \qquad p_{1,n}(x) = nx(1-x)^{n-1},$$

and therefore

$$\frac{x^2}{\phi^2(x)}\, p_{0,n}(x) = (1-x)^{n-2}, \qquad \frac{x}{n\phi^2(x)}\, p_{1,n}(x) = (1-x)^{n-3}.$$

Thus (16) holds clearly for $0 \leqslant x \leqslant 1/n$. This completes the proof.

4.4. *Baskakov Operators*

The Baskakov operator $V_n(f; x)$ is defined as follows.

$$V_n(f; x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k(1+x)^{-n-k} f\left(\frac{k}{n}\right) \qquad (x \in [0, \infty)),$$

$$W(n, x, u) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k(1+x)^{-n-k}\, \delta\left(u - \frac{k}{n}\right).$$

Similarly, we obtain the corresponding theorem for $V_n(f; x)$.

THEOREM 5 (Becker [3]).

$$\text{G.App.}[V_n] = [\{f \mid f \in \text{Lip}_2\, \alpha\}, n^{-\alpha/2}, x(1+x), [0, \infty)].$$

Proof. For $\phi(x) = x(1+x)$ and $p_{k,n}(x) = \binom{n+k-1}{k} x^k(1+x)^{-n-k}$ we have

$$p_{0,n}(x) = (1+x)^{-n}, \qquad p_{1,n}(x) = nx(1+x)^{-n-1},$$

and thus

$$\frac{x^2}{\phi^2(x)}\, p_{0,n}(x) = (1+x)^{-n-2}, \qquad \frac{x}{n\phi^2(x)}\, p_{1,n}(x) = (1+x)^{-n-3}.$$

Therefore (16) is valid for $0 \leqslant x \leqslant 1/n$.

4.5. *Operators by Ismail and May* [8]

The operator $T_n(f; x)$ introduced by Ismail and May is

$$T_n(f; x) = \frac{2^{n-2} n}{\pi \Gamma(n)} (1 + x^2)^{-n/2} \int_{-\infty}^{\infty} \exp(nu \arctan x) \times \left| \Gamma\left(\frac{n}{2} + i \frac{nu}{2}\right) \right|^2 f(u)\, du,$$

$$W(n, x, u) = \frac{2^{n-2} n}{\pi \Gamma(n)} (1 + x^2)^{-n/2} \exp(nu \arctan x) \times \left| \Gamma\left(\frac{n}{2} + i \frac{nu}{2}\right) \right|^2 \qquad (x \in (-\infty, \infty)).$$

Since $\phi(x) = x^2 + 1$ satisfies (9) in Remark 2, we have the following theorem for $T_n(f; x)$.

THEOREM 6.

$$\text{G.App.}[T_n] = [\{f \mid f \in \text{Lip}_2\, \alpha\}, n^{-\alpha/2}, x^2 + 1, (-\infty, \infty)].$$

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